26/10/23

MATH4030 Tutorial

Announcements:

- Midterms aim to return next week.
- HW3 due 30/10.

Recall: S is an orientable surface. Can pick some orientation $N \rightsquigarrow$ for $p \in S$, $S_p(v) = -\frac{d}{dt}N$.

At $p \in S$, can diagonalize the shape operator

$$S_p = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}.$$

(depending on p.) w/ eigenvectors u_1, u_2.

Gaussian curvature $K(p) = \det S_p = k_1 k_2$.

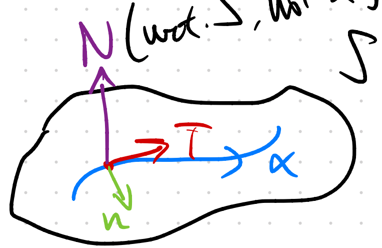
Mean curvature $H(p) = \frac{1}{2} \operatorname{tr} S_p = \frac{1}{2}(k_1 + k_2)$.

k_1, k_2 are principal curvatures

u_1, u_2 are principal directions.

Sps we have a curve α in S.

$\alpha' = T$, N is unit normal wrt. S.

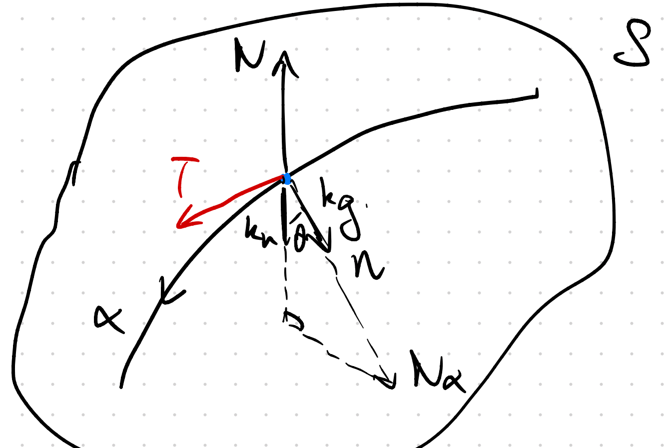


let $n = N \times T$.

Then $\{T, n, N\}$ is positively oriented and we write T' in the basis $\{n, N\}$.

$$T' = k_g n + k_n N.$$

geodesic curvature   normal curvature.



$k_n = \kappa \langle N_\alpha, N \rangle = \kappa \cos\theta$. where θ is the angle between N_α, N.

normal vector w.r.t. α.

let $v \in T_pS$, $\alpha'(0) = v$. Then we write $v = u_1 \cos\varphi + u_2 \sin\varphi$.

Euler's Formula: $k_n = \mathrm{II}_p(v,v) = \langle S_p(v), v\rangle$

$$= \langle S_p(u_1\cos\varphi + u_2\sin\varphi),\ u_1\cos\varphi + u_2\sin\varphi\rangle$$

$$= \langle k_1\cos\varphi\, u_1 + k_2\sin\varphi\, u_2,\ \cos\varphi\, u_1 + \sin\varphi\, u_2\rangle$$

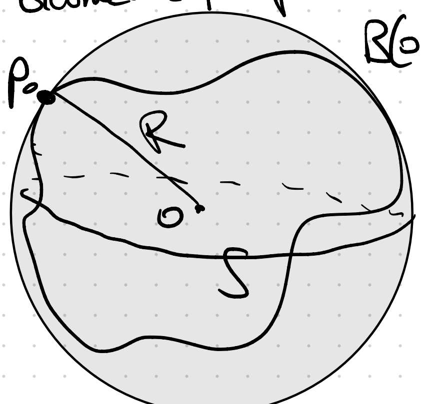
$$= k_1\cos^2\varphi + k_2\sin^2\varphi.$$

Types of Points: $p \in S$. p is called

- Elliptic if $K(p) > 0$. ($k_1, k_2 \neq 0$, same sign, like points on a sphere).
- Hyperbolic if $K(p) < 0$ ($k_1, k_2 \neq 0$, different sign)
- Parabolic if $K(p) = 0$. (one of $k_1, k_2 = 0$)
- Planar if $S_p = 0$. ($k_1 = k_2 = 0$)
- Umbilical if $k_1 = k_2$. (including the planar case).
  (in $\mathbb{R}^3$)

Q1: Show that a surface that is compact has at least one elliptic point.

Pf: "Geometric proof"



$B(0,R)$ Since S is cpt, can enclose it in a sphere $B(0,R)$ and we shrink R until there is at least one point of tangency between S and $B(0,R)$, say p_0.

at p_0, $K_S \geq K_B > 0$.

"Analytic Proof": Since S is cpt, the function $f: S \to \mathbb{R}$ by $f(p) = \|p\|^2$ attains max at some point, say p_0 on S. Let $\alpha(t)$ be a curve on S s.t. $\alpha(0) = p_0$.

Then by maximum, $\frac{d}{dt} f(\alpha(t))\Big|_{t=0} = 0 \Rightarrow 0 = 2\alpha'(0) \cdot \alpha(0)$.

$$\frac{d^2}{dt^2} f(\alpha(t))\Big|_{t=0} \leq 0. \Rightarrow \alpha''(0) \cdot \alpha(0) + \alpha'(0) \cdot \alpha'(0) \leq 0.$$

N at p_0 can be written as $\frac{\alpha(0)}{|\alpha(0)|}$.

Then $\alpha''(0) \cdot \alpha(0) + \alpha'(0) \cdot \alpha'(0) = \alpha''(0) \cdot N |\alpha(0)| + \alpha'(0) \cdot \alpha'(0)$

b/c. at 0,
$\langle \alpha'', N \rangle = -\langle \alpha', dN \rangle$.
$$= |\alpha(0)| \langle -dN(\alpha'(0)), \alpha'(0) \rangle + \alpha'(0) \cdot \alpha'(0)$$
$$= |\alpha(0)| \langle S_p(\alpha'(0)), \alpha'(0) \rangle + \underbrace{\alpha'(0) \cdot \alpha'(0)}_{=1 \text{ : arc-length param.}} \leq 0.$$

In particular, let $\alpha_i(0) = p$, $\alpha_i'(0) = u_i$.
where $u_i(0)$ is the i^{th} principal direction at p_0 for $i = 1, 2$.
and we get $k_i |u_i(0)| + 1 \leq 0. \Rightarrow k_i \leq \frac{-1}{|u_i(0)|} < 0$ for $i = 1, 2$. i.e. $K(p_0) > 0$. //

Q2: Show that there exist no compact minimal surfaces in $\mathbb{R}^3$: $k_1 = k_1(p)$, $k_2 = k_2(p)$.
$(H \equiv 0)$.

Pf: Suppose there is one, say S. Then at each point of S, $H = \frac{1}{2}(k_1 + k_2) = 0$

$\Rightarrow$ k_1, k_2 have opposite signs, or $k_1, k_2 = 0$.

$\Rightarrow$ S has no elliptic points (only planar or hyperbolic or parabolic points).

which contradicts Q1 above. //